Kleene Algebra with Observations

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CONCUR 2019

Introduction

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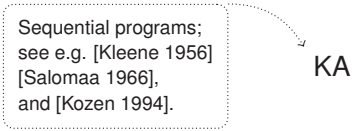

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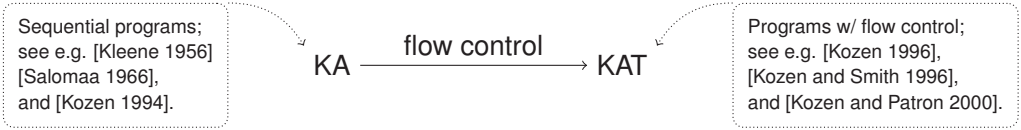

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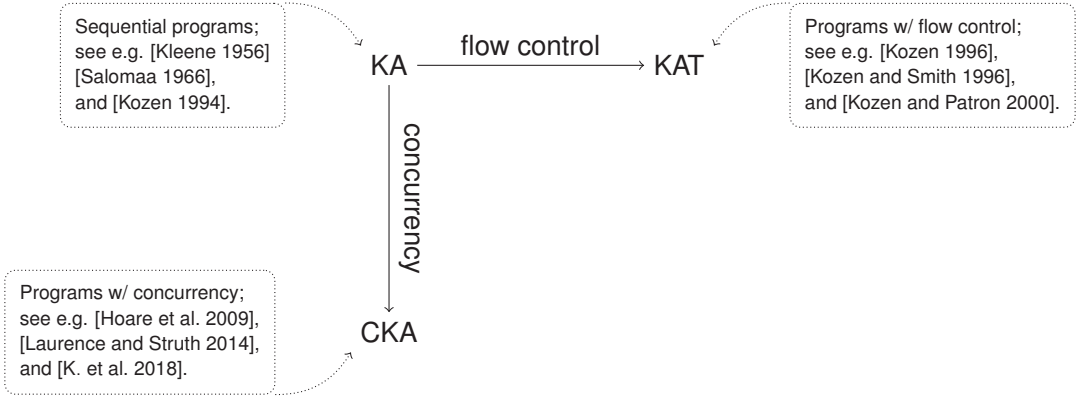

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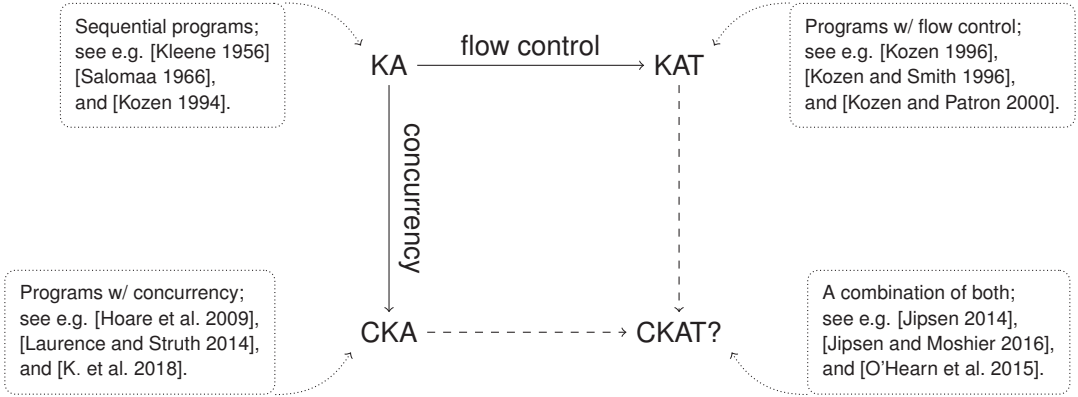

Introduction

$$e, f ::= p \mid 0 \mid 1 \mid a \in \Sigma \mid e + f \mid e \cdot f \mid e^* \mid e \parallel f$$

$$p, q ::= \bot \mid \top \mid t \in T \mid p \vee q \mid p \wedge q \mid \overline{p}$$

$$e, f ::= \underbrace{p \mid 0 \mid 1 \mid a \in \Sigma \mid e + f \mid e \cdot f \mid e^*}_{\text{KAT terms}} \mid e \parallel f$$

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Introduction

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Introduction

$$e + 0 \equiv e \qquad e + e \equiv e \qquad e + f \equiv f + e \qquad e + (f + g) \equiv (e + f) + g$$

$$e \cdot 1 \equiv e \qquad e \equiv 1 \cdot e \qquad e \cdot 0 \equiv 0 \qquad 0 \equiv 0 \cdot e \qquad e \cdot (f \cdot g) \equiv (e \cdot f) \cdot g$$

$$e \cdot (f + g) \equiv e \cdot f + e \cdot g \qquad 1 + e \cdot e^\star \equiv e^\star \qquad e + f \cdot g \leqq g \implies f^\star \cdot e \leqq g$$

$$(e + f) \cdot g \equiv e \cdot g + f \cdot g \qquad 1 + e^\star \cdot e \equiv e^\star \qquad e + f \cdot g \leqq f \implies e \cdot g^\star \leqq f$$

$$e \leqq f \iff e + f \equiv f$$

Introduction

$p \vee \bot \equiv p$ $\quad p \vee q \equiv q \vee p$ $\quad p \vee \overline{p} \equiv \top$ $\quad p \vee (q \vee r) \equiv (p \vee q) \vee r$

$p \wedge \top \equiv p$ $\quad p \wedge q \equiv q \wedge p$ $\quad p \wedge \overline{p} \equiv \bot$ $\quad p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $\quad p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee q \equiv p + q$ $\quad p \wedge q \equiv p \cdot q$ $\quad \top \equiv 1$ $\quad \bot \equiv 0$

Introduction

$$e \parallel f \equiv f \parallel e \qquad e \parallel 1 \equiv e \qquad e \parallel 0 \equiv 0 \qquad (e + f) \parallel g \equiv e \parallel g + f \parallel g$$

$$e \parallel (f \parallel g) \equiv (e \parallel f) \parallel g \qquad (e \parallel f) \cdot (g \parallel h) \leqq (e \cdot g) \parallel (f \cdot h)$$

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$$e \parallel (f \parallel g) \equiv (e \parallel f) \parallel g \qquad (e \parallel f) \cdot (g \parallel h) \leqq (e \cdot g) \parallel (f \cdot h)$$

In particular: $e \cdot f \leqq e \parallel f$.

The problem with CKAT

$$p \cdot e \cdot \overline{p} \leqq (p \parallel e) \cdot \overline{p}$$

The problem with CKAT

$$
\begin{aligned}
p \cdot e \cdot \overline{p} &\leqq (p \parallel e) \cdot \overline{p} \\
&\equiv (p \parallel e) \cdot (\overline{p} \parallel 1)
\end{aligned}
$$

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The problem with CKAT

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Drop $e \parallel 0 \equiv 0$?

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Drop $e \parallel 0 \equiv 0$?

Drop exchange law?

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Non-congruence?

Drop $e \parallel 0 \equiv 0$?

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See also [Bergstra and Ponse 2011].

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A new hope

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Our contribution: a study of the *sequential fragment* (i.e., without $\|$).

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Our contribution: a study of the *sequential fragment* (i.e., without $\|$).

- A semantics $[\![-]\!]$ that sends a term to its *observation language*.

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- Practically feasible decision procedure w.r.t. semantics.

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- A semantics $[\![-]\!]$ that sends a term to its *observation language*.
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- Soundness & completeness: $e \equiv f$ if and only if $[\![e]\!] = [\![f]\!]$.

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These form the foundation for extending to the full syntax.

Semantics

$$\llbracket 0 \rrbracket = \emptyset \qquad \llbracket a \rrbracket = \{a\} \qquad \llbracket e + f \rrbracket = \llbracket e \rrbracket \cup \llbracket f \rrbracket \qquad \llbracket e^\star \rrbracket = \llbracket e \rrbracket^\star$$
$$\llbracket 1 \rrbracket = \{\epsilon\} \qquad \llbracket p \rrbracket = \{\alpha \in \mathsf{At} : \alpha \leqslant p\} \qquad \llbracket e \cdot f \rrbracket = \llbracket e \rrbracket \cdot \llbracket f \rrbracket \,,$$

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$$\alpha \equiv \alpha \wedge \alpha \leqq \alpha \cdot \alpha \quad \text{but} \quad \llbracket \alpha \rrbracket \not\subseteq \llbracket \alpha \cdot \alpha \rrbracket$$

Semantics

$$\frac{w, x \in (\Sigma \cup \mathsf{At})^* \qquad \alpha \in \mathsf{At}}{w\alpha x \preceq w\alpha\alpha x}$$

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Let $[\![-]\!]^{\preceq}$ be the $\preceq$-downclosure of $[\![-]\!]$; then

Lemma (Soundness)

If $e \equiv f$, then $[\![e]\!]^{\preceq} = [\![f]\!]^{\preceq}$.

Decidability

Can we decide whether $[\![e]\!]^{\preceq} = [\![f]\!]^{\preceq}$?

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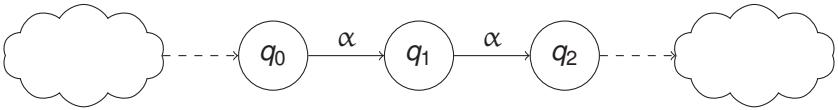
1. Translate e to automaton A_e such that $L(A_e) = [\![e]\!]^{\preceq}$.
2. Translate f to automaton A_f such that $L(A_f) = [\![f]\!]^{\preceq}$.
3. Decide whether $L(A_e) = L(A_f)$.

Decidability



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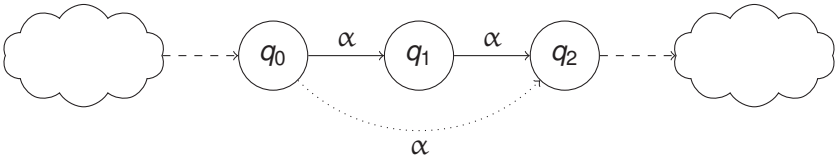

Decidability

$$\epsilon(0) = 0 \qquad \epsilon(e + f) = \epsilon(e) \vee \epsilon(f)$$
$$\epsilon(1) = 1 \qquad \epsilon(e \cdot f) = \epsilon(e) \wedge \epsilon(f)$$
$$\epsilon(a) = 0 \qquad \epsilon(e^*) = 1$$
$$\epsilon(p) = 0$$

See also [Brzozowski 1964].

Decidability

$$\delta(0, a) = \delta(1, a) = \emptyset \qquad \delta(e + f, a) = \delta(e, a) \cup \delta(f, a)$$
$$\delta(a, a') = \{1 : a = a'\} \qquad \delta(e \cdot f, a) = \{e' \cdot f : e' \in \delta(e, a)\} \cup \Delta(e, f, a)$$
$$\delta(p, a) = \emptyset \qquad \delta(e^\star, a) = \{e' \cdot e^\star : e' \in \delta(e, a)\},$$

$$\Delta(e, f, a) = \begin{cases} \delta(f, a) & \epsilon(e) = 1 \\ \emptyset & \epsilon(e) = 0 \end{cases}$$

See also [Antimirov 1996].

Decidability

$$\zeta(0,\alpha) = \delta(1,\alpha) = \emptyset \qquad \zeta(e+f,\alpha) = \zeta(e,\alpha) \cup \zeta(f,\alpha)$$
$$\zeta(a,\alpha) = \emptyset \qquad \zeta(e \cdot f,\alpha) = \{e' \cdot f : e' \in \zeta(e,\alpha)\} \cup Z(e,f,\alpha)$$
$$\zeta(p,\alpha) = \{1 : \alpha \leqslant p\} \qquad \zeta(e^\star,\alpha) = \{e' \cdot e^\star : e' \in \zeta(e,\alpha)\}\,,$$

$$Z(e,f,\alpha) = \begin{cases} \zeta(f,\alpha) & \epsilon(e') = 1 \text{ for some } e' \in \zeta(e,\alpha) \cup \{e\} \\ \emptyset & \text{otherwise} \end{cases}$$

Decidability

Let $\rho(e)$ be the expressions reached through δ or ζ from e.

Lemma

For any e, we have that $A_e = \langle \rho(e), \delta \cup \zeta, \epsilon \rangle$ is a finite non-deterministic automaton.

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Lemma

For any e, we have that $L(A_e) = [\![e]\!]^{\preceq}$.

Theorem

Given e, f, we can decide whether $[\![e]\!]^{\preceq} = [\![f]\!]^{\preceq}$.

Axiomatisation

UCL

An expression e is ...

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- ... *atomic* if its Boolean subterms are atoms.

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An expression e is ...

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Lemma

Let e and f be atomic and closed. If $[\![e]\!]^{\preceq} = [\![f]\!]^{\preceq}$, then $e \equiv f$.

Axiomatisation

Lemma

For every e, we can construct an atomic and closed $\hat{e}$ such that $e \equiv \hat{e}$.

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Proof sketch.

Idea: convert the automaton for e back into an expression.

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Formally, this is the least x_e that solves the system generated by:

$$\epsilon(e) + \sum_{e' \in \delta(e,a)} a \cdot x_{e'} + \sum_{e' \in \zeta(e,\alpha)} \alpha \cdot x_{e'} \leqq x_e$$

Axiomatisation

UCL

Theorem

If $[\![e]\!]^{\preceq} = [\![f]\!]^{\preceq}$, *then* $e \equiv f$.

Axiomatisation

UCL

Theorem

If $[\![e]\!]^{\preceq} = [\![f]\!]^{\preceq}$*, then* $e \equiv f$*.*

Proof.

First, note that

$$[\![\hat{e}]\!] = [\![\hat{e}]\!]^{\preceq} = [\![e]\!]^{\preceq} = [\![f]\!]^{\preceq} = [\![\hat{f}]\!]^{\preceq} = [\![\hat{f}]\!]$$

Axiomatisation

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If $[\![e]\!]^{\preceq} = [\![f]\!]^{\preceq}$*, then* $e \equiv f$*.*

Proof.

First, note that

$$[\![\hat{e}]\!] = [\![\hat{e}]\!]^{\preceq} = [\![e]\!]^{\preceq} = [\![f]\!]^{\preceq} = [\![\hat{f}]\!]^{\preceq} = [\![\hat{f}]\!]$$

Since $\hat{e}$ and $\hat{f}$ are atomic and closed, we conclude

$$e \equiv \hat{e} \equiv \hat{f} \equiv f$$

□

Future work

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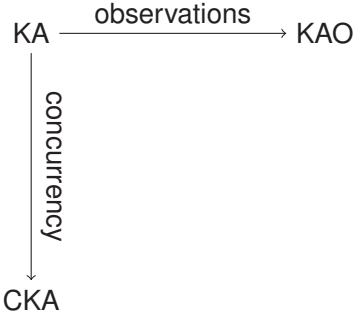
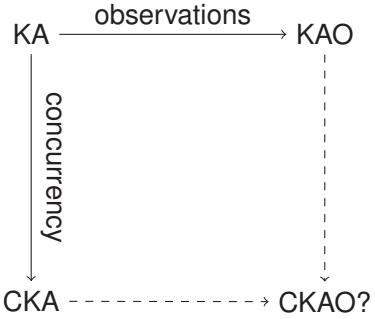


Future work



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2. Isolation: some form of persistence of observations?
3. Optimise decision procedure; exploit transitive property?
4. Possible applications in weak memory.

Thank you for your attention

https://coneco-project.org

Extended paper: https://arxiv.org/abs/1811.10401.

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- "Puzzle" by BlueTip design.
- "Scales" by jai.
- "The Scream" by Jonah Bethlehem.
- "Crossroads Decision" by Anna Sophie.
- "Biking" by tulpahn.
- "Chicken" by BomSymbols.

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