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Institute for Logic, Language and Computation

Completeness and the FMP for KA, revisited

Tobias Kappé

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Some context

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- What can we (not) prove using these laws?
- When is something not true by just the laws of KA?

Definition (Kleene algebra)

A *Kleene algebra* is a tuple $(K, +, \cdot, ^*, 0, 1)$ where

- $(K, +, \cdot, 0, 1)$ is an idempotent semiring
- The operator * additionally satisfies

$$1 + x \cdot x^* = x^* \qquad\qquad x + y \cdot z \leq z \implies y^* \cdot x \leq z$$

Here, $x \leq y$ is a shorthand for $x + y = y$.

Kleene algebra

Expressions and equations

Definition

Fix an alphabet Σ. Exp is the set of *regular expressions*, generated by

$$e, f ::= 0 \mid 1 \mid \mathtt{a} \in \Sigma \mid e + f \mid e \cdot f \mid e^*$$

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Definition

Given a KA $(K, +, \cdot, {}^*, 0, 1)$ and $h : \Sigma \to K$, we define $\hat{h} : \mathrm{Exp} \to K$ by

$$\hat{h}(0) = 0 \qquad \hat{h}(e + f) = \hat{h}(e) + \hat{h}(f)$$
$$\hat{h}(1) = 1 \qquad \hat{h}(e \cdot f) = \hat{h}(e) \cdot \hat{h}(f)$$
$$\hat{h}(\mathtt{a}) = h(\mathtt{a}) \qquad \hat{h}(e^*) = \hat{h}(e)^*$$

Let $e, f \in \mathrm{Exp}$; we write $K \models e = f$ when $\hat{h}(e) = \hat{h}(f)$ for all h.

Kleene algebra

Languages

Fix a (finite) set of *letters* Σ.

Example (KA of languages)

The KA of *languages over* Σ is given by $(\mathcal{P}(\Sigma^*), \cup, \cdot, {}^*, \emptyset, \{\epsilon\})$, where

- $\mathcal{P}(\Sigma^*)$ is the set of sets of words (*languages*);
- $\cdot$ is pointwise concatenation, i.e., $L \cdot K = \{wx : w \in L, x \in K\}$;
- * is the Kleene star, i.e., $L^* = \{w_1 \cdots w_n : w_1, \ldots, w_n \in L\}$;
- ϵ is the empty word.

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Fact: $\mathcal{P}(\Sigma^*) \models e = f$ when e and f denote the same regular language.

Fix a (not necessarily finite) set of *states* S.

Example (KA of relations)

The KA of *relations over* S is given by $(\mathcal{P}(S \times S), \cup, \circ, {}^*, \emptyset, \Delta)$, where

- $\mathcal{P}(S \times S)$ is the set of relations on S;
- $\circ$ is relational composition.
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Fact: $\mathcal{P}(S \times S) \models (\mathsf{a} + 1)^* = \mathsf{a}^*$ because $(R \cup \Delta)^* = R^*$ for all relations R.

Kleene algebra

Model theory

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- $\vdash e = f$ when $e = f$ follows from the axioms of KA.

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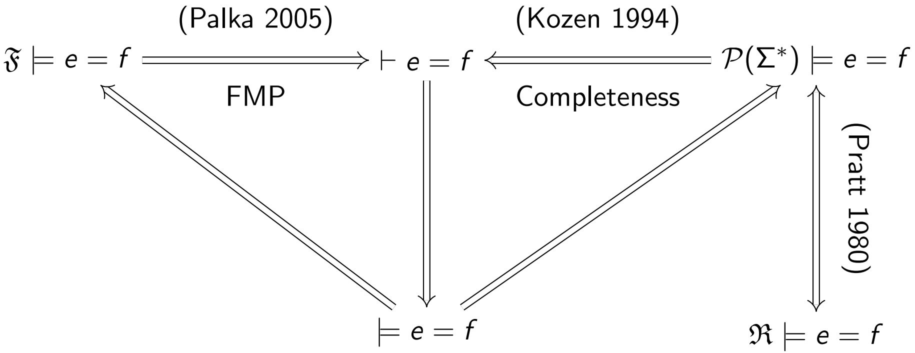
- $\models e = f$ when $K \models e = f$ for every KA K.
- $\mathfrak{R} \models e = f$ when $\mathcal{P}(S \times S) \models e = f$ for all S.
- $\mathfrak{F} \models e = f$ when $K \models e = f$ holds in every finite KA K.

Kleene algebra

Model theory



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Given e, f, we do the following:

1. Turn expressions e, f into a finite automaton A
2. Turn the finite automaton A into a finite monoid M
3. Turn the finite monoid M into a finite KA K

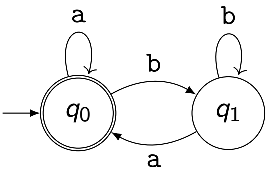
Expressions to automata

Definition

An automaton is a tuple $(Q, \rightarrow, I, F)$ where

- Q is a finite set of *states*; and
- $\rightarrow \subseteq Q \times \Sigma \times Q$ is the *transition relation*; and
- $I \subseteq Q$ is the set of *initial states*
- $F \subseteq Q$ is the set of *accepting states*

We write $q \xrightarrow{\mathtt{a}} q'$ when $(q, \mathtt{a}, q') \in \rightarrow$.



Expressions to automata

Definition

Let $(Q, \to, F)$ be an automaton. A *solution* is a function $s : Q \to \mathsf{Exp}$ such that

$$\vdash F(q) + \sum_{q \xrightarrow{\mathtt{a}} q'} \mathtt{a} \cdot s(q') \leq s(q)$$

Here, $F(q) = 1$ when $q \in F$ and $F(q) = 0$ otherwise.

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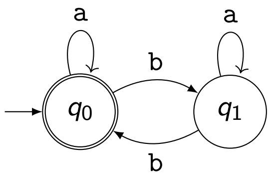
Example

For the automaton on the right, a solution satisfies

$$\vdash 1 + \mathtt{a} \cdot s(q_0) + \mathtt{b} \cdot s(q_1) \leq s(q_0)$$

$$\vdash 0 + \mathtt{a} \cdot s(q_1) + \mathtt{b} \cdot s(q_0) \leq s(q_1)$$



E.g., $s(q_0) = (\mathtt{a} + \mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b})^*$ and $s(q_1) = \mathtt{a}^* \cdot \mathtt{b} \cdot s(q_0)$.

Expressions to automata

Theorem (Kleene 1956; see also Conway 1971)

Every automaton admits a least solution (unique up to equivalence).

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When A is an automaton, we write $A(q)$ for its least solution at q.

Lemma (c.f. Kleene 1956; Antimirov 1996; Kozen 2001; Jacobs 2006)
For every e, we can construct an automaton $A_e = (Q_e, \rightarrow_e, I_e, F_e)$ such that

$$\vdash e = \sum_{q \in I_e} A_e(q)$$

Automata to monoids

Let $A = (Q, \rightarrow, I, F)$ be an automaton.

Definition (Transition monoid; McNaughton and Papert 1968)

$(M_A, \circ, \Delta)$ is a monoid, where $M_A = \{\xrightarrow{\mathtt{a}_1} \circ \cdots \circ \xrightarrow{\mathtt{a}_n} : \mathtt{a}_1, \ldots, \mathtt{a}_n \in \Sigma\}$.

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Let $R \in M_A$. We write $A[R]$ for the *transition automaton* $(M_A, \to_\circ, \Delta, \{R\})$ where

$$P \xrightarrow{\mathtt{a}}_\circ Q \iff P \circ \xrightarrow{\mathtt{a}} = Q$$

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Lemma (Solving transition automata)

$$\vdash A(q) = \sum_{qRq_f \in F} A[R](\Delta)$$

Monoids to Kleene algebras

Lemma (Palka 2005)

Let $(M, \cdot, 1)$ *be a monoid. Now* $(\mathcal{P}(M), \cup, \otimes, {}^{\circledast}, \emptyset, \{1\})$ *is a KA, where*

$$T \otimes U = \{t \cdot u : t \in T \wedge u \in U\} \qquad T^{\circledast} = \{t_1 \cdots t_n : t_1, \ldots, t_n \in T\}$$

Monoids to Kleene algebras

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Lemma

Let A *be an automaton, and let* $h : \Sigma \to \mathcal{P}(M_A)$ *where* $h(\mathtt{a}) = \{\xrightarrow{\mathtt{a}}\}$*. Now*

$$R \in \hat{h}(A(q)) \iff q \; R \; q_f \in F$$

Putting it all together

In the sequel, fix $e, f \in \mathrm{Exp}$, and:

- Let $A_{e,f} = (Q_{e,f}, \rightarrow_{e,f}, I_{e,f}, F_{e,f})$ be the disjoint union of A_e and A_f.
- Let $M_{e,f} = (M_{A_{e,f}}, \circ, \Delta)$ be the monoid of $A_{e,f}$.

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Lemma (Normal form)

Let $e, f \in \mathsf{Exp}$ and $h : \Sigma \to \mathcal{P}(M_{e,f})$ be given by $h(\mathtt{a}) = \{\xrightarrow{\mathtt{a}}_{e,f}\}$. The following hold:

$$\vdash e = \sum_{R \in \hat{h}(e)} A_{e,f}[R](\Delta) \qquad\qquad \vdash f = \sum_{R \in \hat{h}(f)} A_{e,f}[R](\Delta)$$

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Finite model property

Theorem (Finite model property)

If $\mathfrak{F} \models e = f$ *then* $\vdash e = f$.

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Proof.

$\mathcal{P}(M_{e,f})$ is a finite KA; hence $\mathcal{P}(M_{e,f}) \models e = f$, i.e., $\hat{h}(e) = \hat{h}(f)$. But then:

$$\vdash e = \sum_{R \in \hat{h}(e)} A_{e,f}[R](\Delta) = \sum_{R \in \hat{h}(f)} A_{e,f}[R](\Delta) = f \qquad \square$$

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Let $L : \Sigma \to \mathcal{P}(\Sigma^*)$ be given by $L(\mathtt{a}) = \{\mathtt{a}\}$.

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Completeness

Theorem (Completeness)

If $\mathcal{P}(\Sigma^*) \models e = f$ *then* $\vdash e = f$*.*

Proof.

Let $L : \Sigma \to \mathcal{P}(\Sigma^*)$ be given by $L(\mathtt{a}) = \{\mathtt{a}\}$.

We can show that $\hat{h}(e) = \{\xrightarrow{\mathtt{a}_1}_{e,f} \circ \cdots \circ \xrightarrow{\mathtt{a}_n}_{e,f} : \mathtt{a}_1 \cdots \mathtt{a}_n \in \hat{L}(e)\}$, and similarly for f.

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We can show that $\hat{h}(e) = \{\xrightarrow{\mathtt{a}_1}_{e,f} \circ \cdots \circ \xrightarrow{\mathtt{a}_n}_{e,f} : \mathtt{a}_1 \cdots \mathtt{a}_n \in \hat{L}(e)\}$, and similarly for f.

If $\mathcal{P}(\Sigma^*) \models e = f$, then $\hat{L}(e) = \hat{L}(e)$, so $\hat{h}(e) = \hat{h}(f)$. The rest proceeds as before. $\square$

Coq formalization

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- Some concepts are encoded differently; ideas remain the same.

Open questions

- Can we apply these ideas to *guarded Kleene algebra with tests*?

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- Does KA have a *finite relational model property*?
- Do these techniques extend to *KA with hypotheses*?
- Is there a representation theorem or duality for KA?

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