Cornell University

# Associativity, monads, and diagrams

Tobias Kappé

JNF Group Score — March 29, 2021

# Associativity

$\odot : X \times X \to X$

$$x_1 \odot (x_2 \odot x_3) = (x_1 \odot x_2) \odot x_3$$

$$\begin{array}{ccccc}
(X \times X) \times X & & \xrightarrow{\alpha} & & X \times (X \times X) \\
\odot \times \mathrm{id} \big\downarrow & & & & \big\downarrow \mathrm{id} \times \odot \\
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# Adding a monad

$$\odot : X \times X \to TX \qquad \hat{\odot} : TX \times TX \to TX$$

$$TX \times TX \xrightarrow{\psi} T(X \times X) \xrightarrow{T\odot} T^2X \xrightarrow{\mu} TX$$

"pair of sets" "set of pairs" "set of sets" "set"

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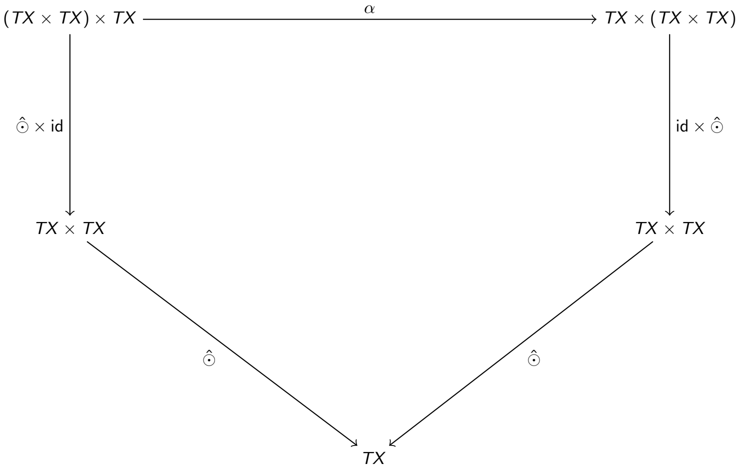
# Lifted associativity



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\begin{array}{ccccc}
(TX \times TX) \times TX & & \xrightarrow{\quad \alpha \quad} & & TX \times (TX \times TX) \\
\downarrow{\scriptstyle \psi \times \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id} \times \psi} \\
T(X \times X) \times TX & & & & TX \times T(X \times X) \\
\downarrow{\scriptstyle T\odot \times \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id} \times T\odot} \\
T^2X \times T^2X & & & & T^2X \times T^2X \\
\downarrow{\scriptstyle \mu \times \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id} \times \mu} \\
TX \times TX & & & & TX \times TX \\
\downarrow{\scriptstyle \psi} & & & & \downarrow{\scriptstyle \psi} \\
T(X \times X) & & & & T(X \times X) \\
\downarrow{\scriptstyle T\odot} & & & & \downarrow{\scriptstyle T\odot} \\
T^2X & & & & T^2X \\
& \searrow{\scriptstyle \mu} & & \swarrow{\scriptstyle \mu} & \\
& & TX & &
\end{array}
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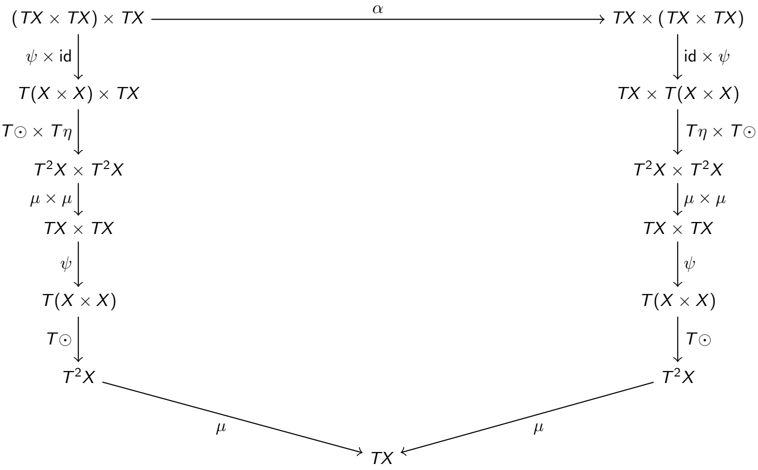

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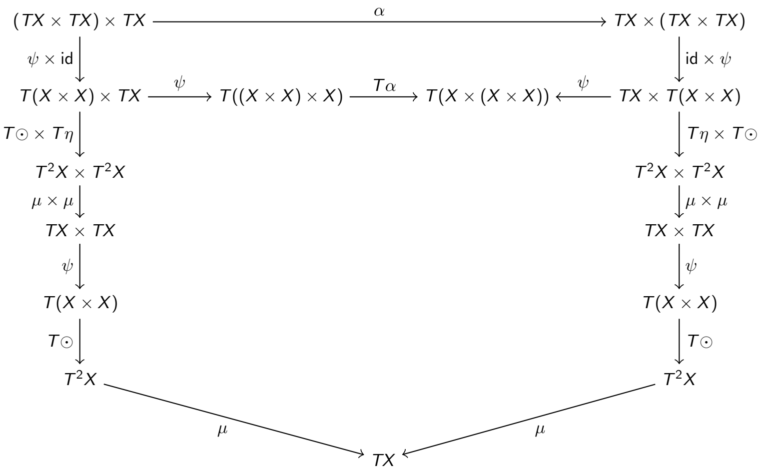

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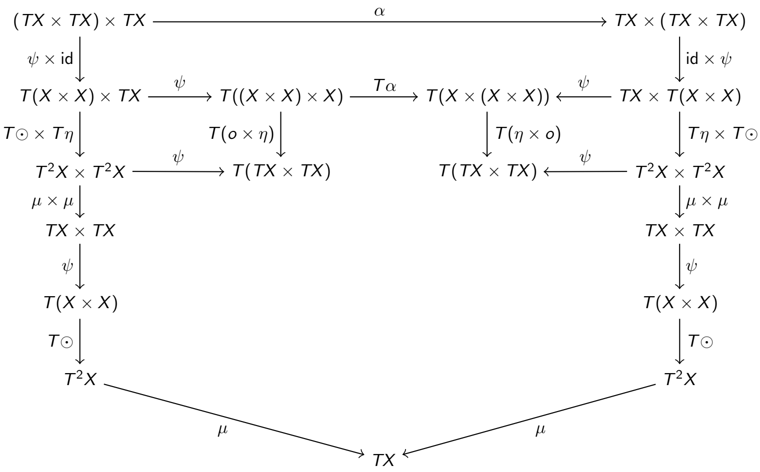

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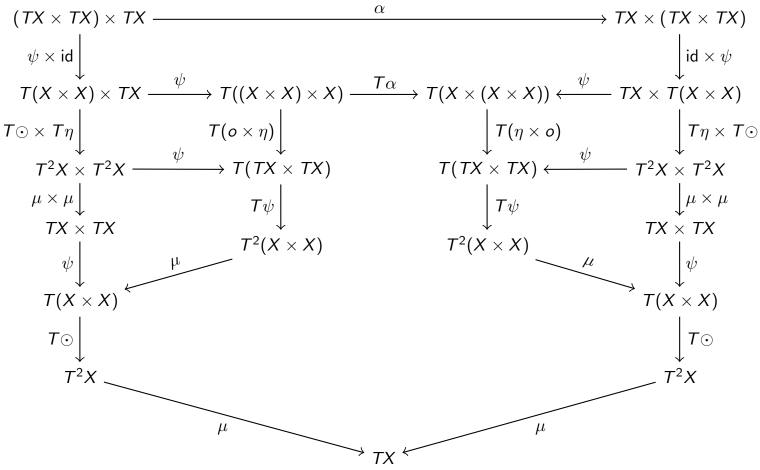

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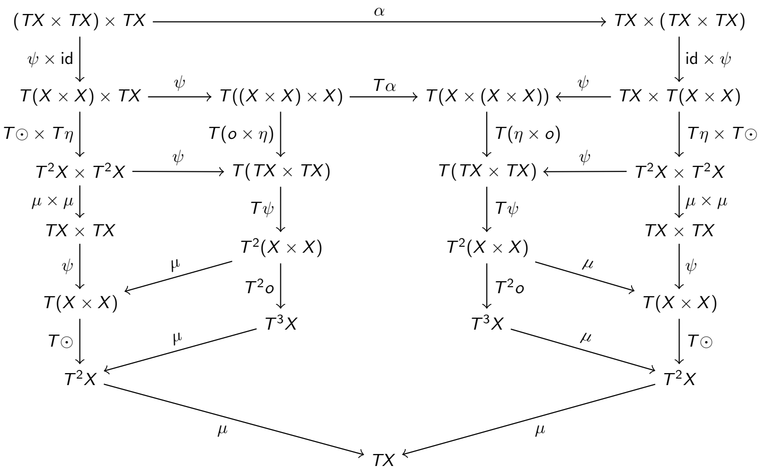

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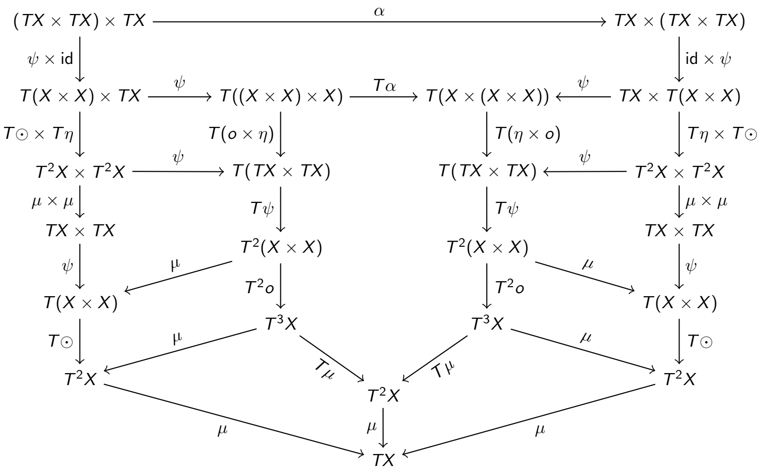

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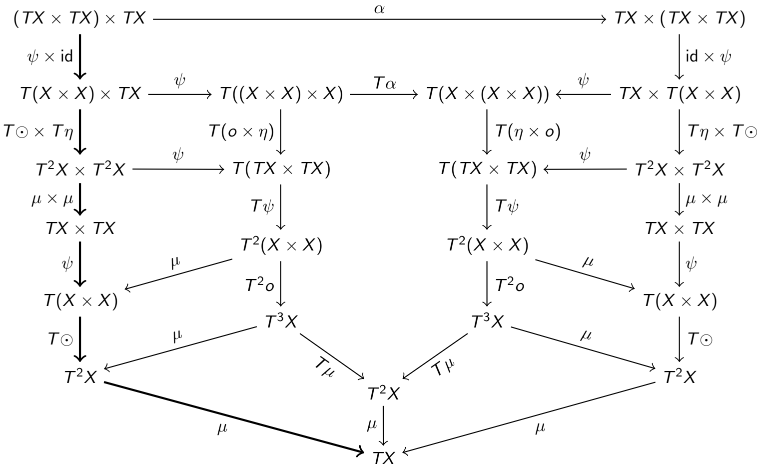

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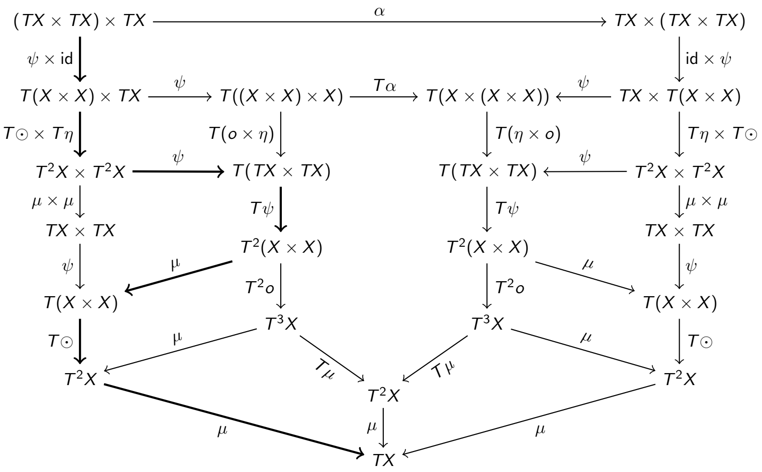

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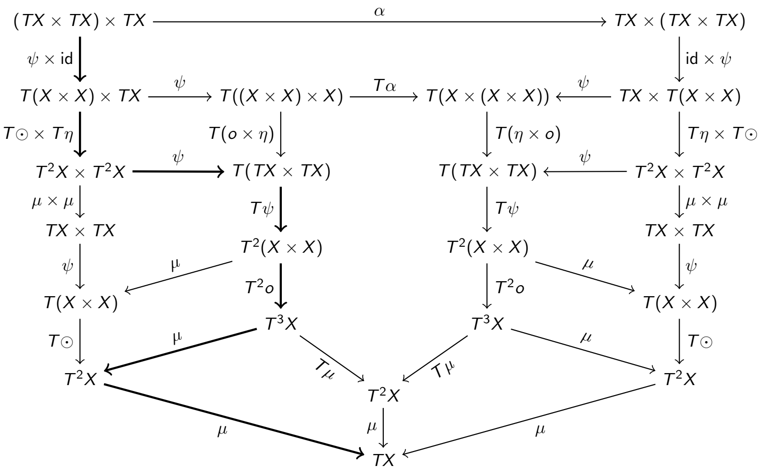

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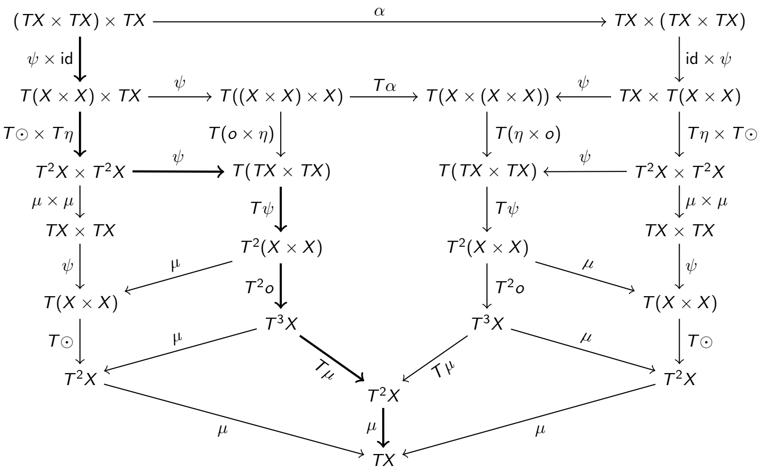

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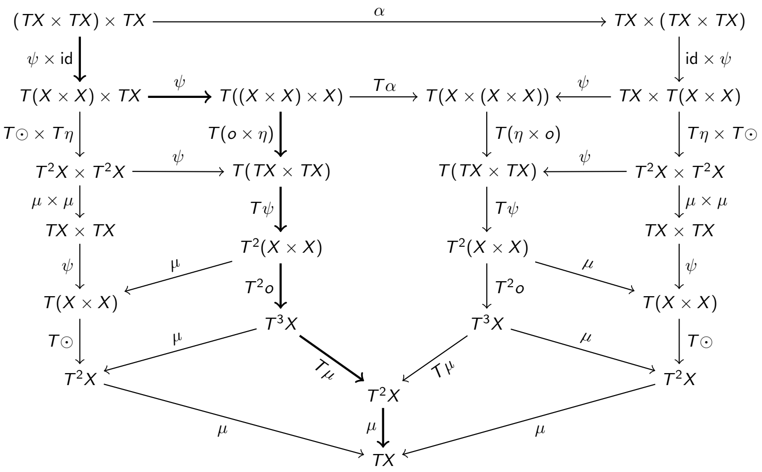

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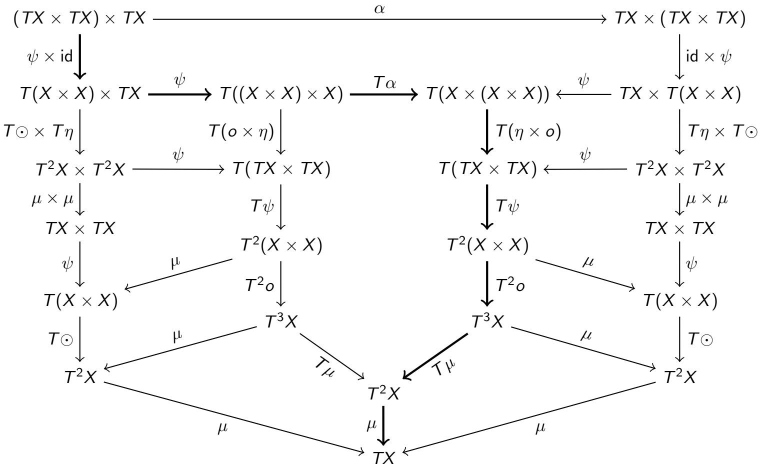

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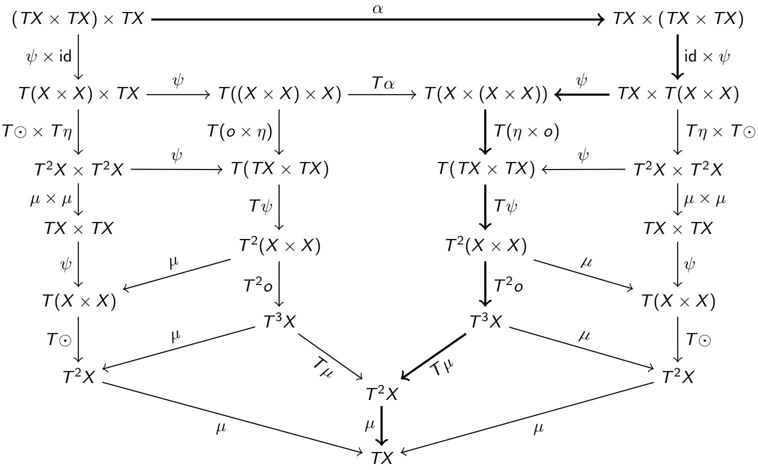

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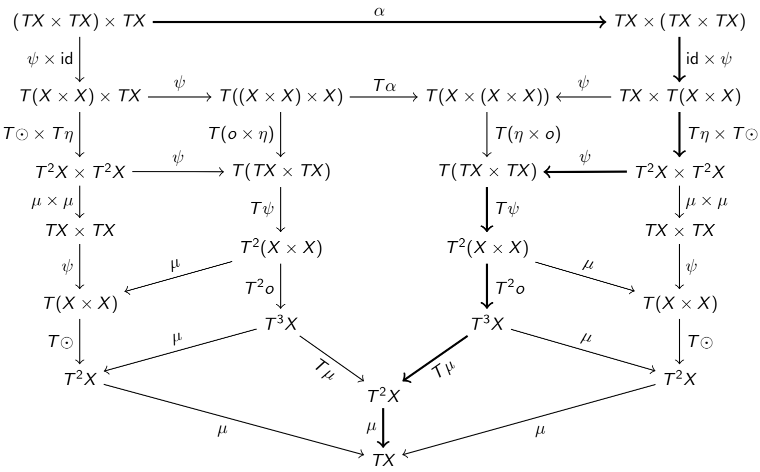

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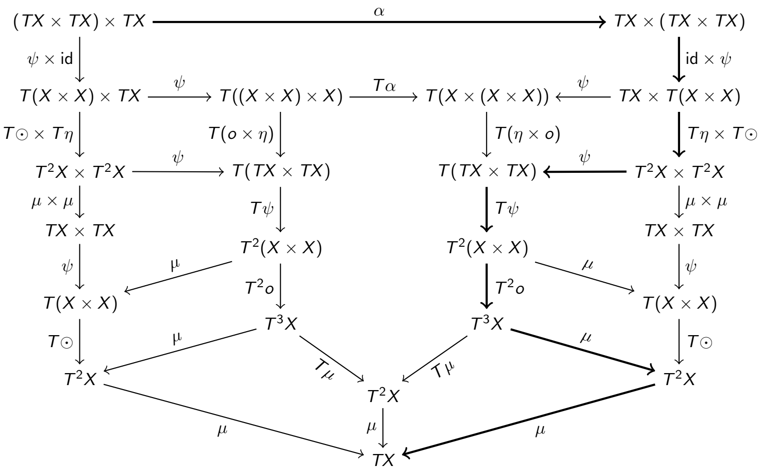

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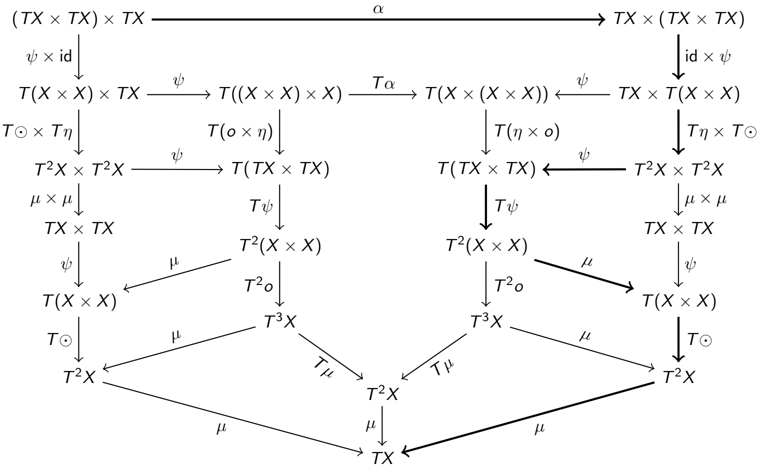

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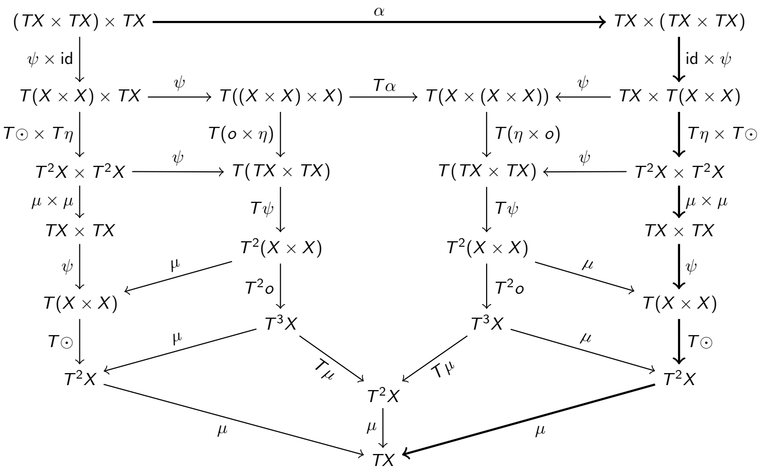

# The inner diagram

**Theorem**

*$\hat{\odot}$ is associative iff the following commutes:*

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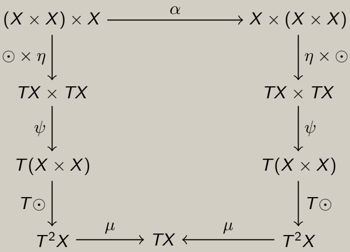


If $T$ is the powerset monad:

$$\frac{x_{1,2} \in x_1 \odot x_2 \qquad x \in x_{1,2} \odot x_3}{\exists x_{2,3} \in x_2 \odot x_3.\ x \in x_1 \odot x_{2,3}}$$

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Similar conditions for multisets, probability, etc.