Completeness for Concurrent Kleene Algebra

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NII Logic Seminar

Introduction

Kleene Algebra models *program flow*.

- abort (0) and skip (1)
- atomic actions (a, b, ...)
- non-deterministic choice ($+$)
- sequential composition ($\cdot$)
- indefinite repetition (*)

$$(e + f)^* \equiv_{\mathsf{KA}} e^* \cdot (f \cdot e^*)^*$$

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$$(a \cdot b) \parallel (c \cdot d)$$

Concurrent KA[1] adds *parallel composition* ($\parallel$)

[1] Hoare, Möller, Struth, and Wehrman 2009

Introduction

KA is well-studied:

- Decision procedures [Hopcroft and Karp 1971; Bonchi and Pous 2013]
- Coalgebra, automata [Kleene 1956; Brzozowski 1964; Silva 2010]
- Axiomatisation of equivalence [Salomaa 1966; Conway 1971; Kozen 1994]

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CKA is a work in progress:

- Decision procedures [Brunet, Pous, and Struth 2017]
- Coalgebra, automata [K., Brunet, Luttik, Silva, and Zanasi 2017]
- Axiomatisation of equivalence [Gischer 1988; Laurence and Struth 2014]

Introduction

Theorem (Kozen 1994)

The axioms for KA *are complete for equivalence:*

$$e \equiv_{\mathsf{KA}} f \iff [\![e]\!]_{\mathsf{KA}} = [\![f]\!]_{\mathsf{KA}}$$

$[\![-]\!]_{\mathsf{KA}}$ *is the regular language interpretation of e.*

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Question

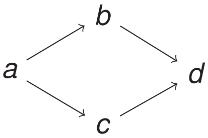
Can we find axioms for CKA *that are complete for equivalence? That is,*

$$e \equiv_{\mathsf{CKA}} f \overset{?}{\iff} [\![e]\!]_{\mathsf{CKA}} = [\![f]\!]_{\mathsf{CKA}}$$

$[\![-]\!]_{\mathsf{CKA}}$ *is a generalized regular language interpretation of e.*

Preliminaries

- Pomset: “word with parallelism”

$a \cdot (b \parallel c) \cdot d =$ 

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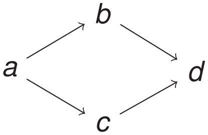

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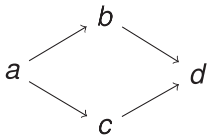

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- Composition lifts:
  - $\mathcal{U} \cdot \mathcal{V} = \{U \cdot V : U \in \mathcal{U}, V \in \mathcal{V}\}$
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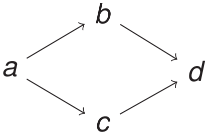
$a \cdot (b \parallel c) \cdot d =$ 

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- Kleene star: $\mathcal{U}^* = \bigcup_{n<\omega} \mathcal{U}^n$

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BKA *semantics* is given by $[\![-]\!]_{\mathsf{BKA}} : \mathcal{T} \to 2^{\mathsf{Pom}_\Sigma}$.

$$\begin{aligned}
[\![0]\!]_{\mathsf{BKA}} &= \emptyset \\
[\![1]\!]_{\mathsf{BKA}} &= \{1\} \\
[\![a]\!]_{\mathsf{BKA}} &= \{a\} \\
[\![e + f]\!]_{\mathsf{BKA}} &= [\![e]\!]_{\mathsf{BKA}} \cup [\![f]\!]_{\mathsf{BKA}} \\
[\![e \cdot f]\!]_{\mathsf{BKA}} &= [\![e]\!]_{\mathsf{BKA}} \cdot [\![f]\!]_{\mathsf{BKA}} \\
[\![e \parallel f]\!]_{\mathsf{BKA}} &= [\![e]\!]_{\mathsf{BKA}} \parallel [\![f]\!]_{\mathsf{BKA}} \\
[\![e^*]\!]_{\mathsf{BKA}} &= [\![e]\!]_{\mathsf{BKA}}^*
\end{aligned}$$

Preliminaries

Axioms for BKA :

$$e + 0 \equiv_{\text{BKA}} e \qquad e \cdot 1 \equiv_{\text{BKA}} e \equiv_{\text{BKA}} 1 \cdot e \qquad e \cdot 0 \equiv_{\text{BKA}} 0 \equiv_{\text{BKA}} 0 \cdot e$$

$$e + e \equiv_{\text{BKA}} e \qquad e + f \equiv_{\text{BKA}} f + e \qquad e + (f + g) \equiv_{\text{BKA}} (f + g) + h$$

$$e \cdot (f \cdot g) \equiv_{\text{BKA}} (e \cdot f) \cdot g \qquad e \cdot (f + g) \equiv_{\text{BKA}} e \cdot f + e \cdot h \qquad (e + f) \cdot g \equiv_{\text{BKA}} e \cdot g + f \cdot g$$

$$1 + e \cdot e^* \equiv_{\text{BKA}} e^* \qquad e \cdot f + g \leqq_{\text{BKA}} f \implies e^* \cdot g \leqq_{\text{BKA}} f$$

$$e \parallel f \equiv_{\text{BKA}} f \parallel e \qquad e \parallel 1 \equiv_{\text{BKA}} e \qquad e \parallel 0 \equiv_{\text{BKA}} 0$$

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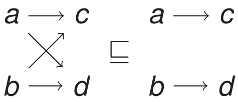

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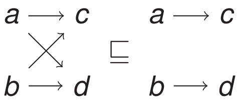

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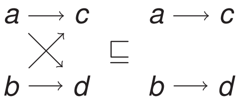

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- *Closure* under pomset subsumption: $\mathcal{U}{\downarrow} = \{U' \sqsubseteq U : U \in \mathcal{U}\}$

$\mathcal{U}{\downarrow}$: all "sequentialisations" of pomsets in $\mathcal{U}$.

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- Axioms to build $\equiv_{\mathrm{CKA}}$: all axioms for $\equiv_{\mathrm{BKA}}$, as well as the *exchange law*:

$$(e \parallel f) \cdot (g \parallel h) \leqq_{\mathrm{CKA}} (e \cdot g) \parallel (f \cdot h)$$

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Lemma (Hoare, Möller, Struth, and Wehrman 2009)

The axioms of CKA are sound for equivalence, i.e.,

$$e \equiv_{\mathrm{CKA}} f \implies [\![e]\!]_{\mathrm{CKA}} = [\![f]\!]_{\mathrm{CKA}}$$

Preliminaries

Theorem (Kozen 1994)

Let M be an n-by-n matrix over $\mathcal{T}$, and $\vec{b}$ an n-dimensional vector over $\mathcal{T}$.

The inequation $M \cdot \vec{x} + \vec{b} \leqq_{\mathsf{KA}} \vec{x}$ admits a unique least solution (with respect to $\leqq_{\mathsf{KA}}$).

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- This “fixpoint” can be constructed *fully syntactically*.
- The same works for BKA and CKA.
- In fact, the solution is the same in both systems!
- We use this as a device to find specific terms later on.

Closure

Definition

Let $e \in \mathcal{T}$; a *closure* of e is a term $e{\downarrow}$ such that

1. $e{\downarrow} \equiv_{\mathsf{CKA}} e$
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Lemma (Laurence & Struth)

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Proof.

Observe that $[\![e{\downarrow}]\!]_{\mathsf{BKA}} = [\![f{\downarrow}]\!]_{\mathsf{BKA}}$, and therefore $e \equiv_{\mathsf{CKA}} e{\downarrow} \equiv_{\mathsf{BKA}} f{\downarrow} \equiv_{\mathsf{CKA}} f$. □

Closure

Lemma

If e, f *have closures* $e{\downarrow}$ *and* $f{\downarrow}$ *respectively, then*

1. $e{\downarrow} + f{\downarrow}$ *is a closure of* $e + f$
2. $e{\downarrow} \cdot f{\downarrow}$ *is a closure of* $e \cdot f$
3. $e{\downarrow}^*$ *is a closure of* e^*

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Induction hypothesis: for $e \in \mathcal{T}$, we assume that:

- If f is a strict subterm of e, we can construct $f{\downarrow}$.
- If $|f| < |e|$ we can construct $f{\downarrow}$.[2]

[2] $|e|$ is the nesting level e w.r.t. $\parallel$

Preclosure

A *preclosure* is almost a closure, but not quite.

Definition

Let $e \in \mathcal{T}$. A *preclosure* of e is a term $\tilde{e} \in \mathcal{T}$ such that

1. $\tilde{e} \equiv_{\mathsf{CKA}} e$.
2. if $U \in [\![e]\!]_{\mathsf{CKA}}$ is non-sequential, then $U \in [\![\tilde{e}]\!]_{\mathsf{BKA}}$

Preclosure

Definition

Let $e \in \mathcal{T}$; Δ_e is the smallest relation on $\mathcal{T}$ such that

$$\frac{}{1 \;\Delta_e\; e} \qquad \frac{}{e \;\Delta_e\; 1} \qquad \frac{\ell \;\Delta_{e_0}\; r}{\ell \;\Delta_{e_1+e_0}\; r} \qquad \frac{\ell \;\Delta_{e_1}\; r}{\ell \;\Delta_{e_0+e_1}\; r} \qquad \frac{\ell \;\Delta_e\; r}{\ell \;\Delta_{e^\star}\; r}$$

$$\frac{\ell \;\Delta_{e_0}\; r \qquad 1 \in [\![e_1]\!]_{\mathsf{CKA}}}{\ell \;\Delta_{e_0 \cdot e_1}\; r} \qquad \frac{\ell \;\Delta_{e_1}\; r \qquad 1 \in [\![e_0]\!]_{\mathsf{CKA}}}{\ell \;\Delta_{e_0 \cdot e_1}\; r} \qquad \frac{\ell_0 \;\Delta_{e_0}\; r_0 \qquad \ell_1 \;\Delta_{e_1}\; r_1}{\ell_0 \parallel \ell_1 \;\Delta_{e_0 \parallel e_1}\; r_0 \parallel r_1}$$

Lemma

Let $V, W \neq 1$, $e \in \mathcal{T}$, and $V \parallel W \in [\![e]\!]_{\mathsf{BKA}}$; there exist $\ell \;\Delta_e\; r$ with $V \in [\![\ell]\!]_{\mathsf{BKA}}$ and $W \in [\![r]\!]_{\mathsf{BKA}}$.

Preclosure

Definition

Let $e, f \in \mathcal{T}$; the term $e \odot f$ is defined as follows:

$$e \odot f \triangleq e \parallel f + \sum_{\substack{\ell \Delta_{e \parallel f} r \\ |\ell|, |r| < |e \parallel f|}} \ell{\downarrow} \parallel r{\downarrow}$$

Lemma

Let $e, f \in \mathcal{T}$; then

1. $e \odot f \equiv_{\mathsf{CKA}} e \parallel f$
2. *if $U \in [\![e \parallel f]\!]_{\mathsf{CKA}}$ is non-sequential, then $U \in [\![e \odot f]\!]_{\mathsf{BKA}}$*

That is, $e \odot f$ is a preclosure of $e \parallel f$.

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For instance: if $e = a \cdot b$ and $f = c \cdot d$:

- $(a \parallel c) \cdot (b \parallel d) \leqq_{\mathsf{CKA}} e \parallel f$ $\qquad (e = a \bullet b, f = c \bullet d)$

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- $c \cdot ((a \cdot b) \parallel d) \leqq_{\text{CKA}} e \parallel f$ $(e = 1 \bullet a \cdot b, f = c \bullet d)$
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Goal: find enough of these terms to cover all pomsets in $[\![e \parallel f]\!]_{\mathsf{CKA}}$.

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Definition

Let $e \in \mathcal{T}$. We define $\nabla_e \subseteq \mathcal{T} \times \mathcal{T}$ as the smallest relation such that

$$\frac{}{1 \nabla_1 1} \qquad \frac{}{a \nabla_a 1} \qquad \frac{}{1 \nabla_a a} \qquad \frac{}{1 \nabla_{e^*} 1} \qquad \frac{\ell \nabla_e r}{\ell \nabla_{e+f} r} \qquad \frac{\ell \nabla_f r}{\ell \nabla_{e+f} r}$$

$$\frac{\ell \nabla_e r}{\ell \nabla_{e \cdot f} r \cdot f} \qquad \frac{\ell \nabla_f r}{e \cdot \ell \nabla_{e \cdot f} r} \qquad \frac{\ell_0 \nabla_e r_0 \qquad \ell_1 \nabla_f r_1}{\ell_0 \parallel \ell_1 \nabla_{e \parallel f} r_0 \parallel r_1} \qquad \frac{\ell \nabla_e r}{e^* \cdot \ell \nabla_{e^*} r \cdot e^*}$$

Lemma

Let $e \in \mathcal{T}$ and $U \cdot V \in [\![e]\!]_{\mathsf{WCKA}}$; there exist $\ell \nabla_e r$ such that $U \in [\![\ell]\!]_{\mathsf{CKA}}$ and $V \in [\![r]\!]_{\mathsf{CKA}}$.

Closure

Suppose that for all $g, h \in \mathcal{T}$, we have that $X_{g \parallel h}$ is a closure of $g \parallel h$.

Then we find

$$e \parallel f + \sum_{\substack{\ell_e \,\nabla_e\, r_e \\ \ell_f \,\nabla_f\, r_f}} (\ell_e \parallel \ell_f) \cdot (r_e \parallel r_f) \leqq_{\mathsf{CKA}} X_{e \parallel f}$$

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For $X_{r_e \parallel r_f}$, we find another inequation, et cetera. . .

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Lemma

Continuing this, we get a finite system of inequations $\langle M, \vec{b} \rangle_{e \parallel f}$.

Closure

Theorem

Let $e \otimes f$ be the least solution to $X_{e \parallel f}$ in $\langle M, \vec{b} \rangle_{e \parallel f}$. Then the following hold:

1. $e \otimes f \equiv_{\mathsf{CKA}} e \parallel f$
2. $[\![e \otimes f]\!]_{\mathsf{BKA}} = [\![e \parallel f]\!]_{\mathsf{CKA}}$

In other words, $e \otimes f$ is a closure of $e \parallel f$.

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Theorem

If $e \in \mathcal{T}$, then we can compute a term $e{\downarrow}$ that is a closure of e.

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Theorem

If $e \in \mathcal{T}$, then we can compute a term $e{\downarrow}$ that is a closure of e.

Corollary

Let $e, f \in \mathcal{T}$ be such that $[\![e]\!]_{\mathsf{CKA}} = [\![f]\!]_{\mathsf{CKA}}$; then $e \equiv_{\mathsf{CKA}} f$.

Conclusion

- Axiomatised equality of *closed, rational pomset languages*.
- Results establishes these as the carrier of the free CKA.
- Extends half of earlier Kleene theorem: terms to pomset automata.
- We also obtain a novel (but inefficient) decision procedure.

Further work

- Explore coalgebraic perspective:
  - Efficient equivalence checking through bisimulation?
  - Can completeness be shown coalgebraically?
- Add “parallel star” operator — closure method does not apply.
- Endgame: lift results to KAT, then NetKAT.

Thank you for your attention

GoNeCo

Implementation: https://doi.org/10.5281/zenodo.926651.

Draft paper: https://arxiv.org/abs/1710.02787.